# Challenging Mathematics Problems for High School Students - Book 1

**Vamshi Jandhyala**

# Preface

Welcome to a journey through the enchanting world of mathematics, a subject that extends far beyond the confines of classrooms and textbooks. This series of problem books is designed to reveal to high school students not just the utility of mathematics, but its inherent beauty, its powerful techniques, and the magic that lies in every equation and theorem.

Mathematics is more than a collection of problems; it is a gateway to critical thinking and creative exploration. It connects us to the great minds of history, from ancient philosophers to modern scientists, all of whom saw in mathematics a language capable of explaining the universe. As you delve into these pages, you will encounter challenges meant to inspire, to push your boundaries, and to unveil the elegance hidden within complex patterns and solutions.

Our goal is not simply to teach you how to solve problems, but to encourage you to appreciate the journey to the solution. Each technique you master is a step closer to understanding the world in a more profound way. Through these problems, we invite you to discover the joy of a well-crafted solution and the satisfaction that comes from unraveling mathematical mysteries.

This series is your invitation to see mathematics not as a hurdle, but as a rich landscape filled with intriguing puzzles that ignite the imagination. Let these books be your guide to a deeper appreciation of the world around you, seen through the lens of mathematical thought.

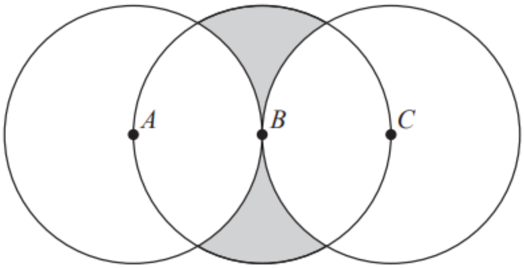
Embark on this adventure with an open mind, and let the beauty of mathematics inspire you at every step.

***Vamshi Jandhyala***

# Problems

**Problem 1**: Let $O(0,0)$ and $A(0,1)$. Suppose a point $B$ is chosen(uniformly) at random on the circle $x^2 + y^2 = 1$. What is the probability that $OAB$ is a triangle whose area is atleast $\frac{1}{4}$?

**Problem 2**: The diagram shows three circles, each of radius 4cm. The centers of the circles are $A$, $B$ and $C$ such that $ABC$ is a straight line and $AB = BC = 4cm$. What is the total area of the two shaded regions. Give your answer in terms of $\pi$.




**Problem 3**: The centre of a circle is the point with coordinates $(-1, 3)$. The point $A$ with coordinates $(6, 8)$ lies on the circle. Find an equation of the tangent to the circle at $A$. Give your answer in terms of $ax + by + c = 0$ where $a, b$ and $c$ are integers.

**Problem 4**: Let $ABCD$ be a square with each side of length 1 unit. If $M$ is the intersection of its diagonals and $P$ is the midpoint of $MB$, what is the square of the length of $AP$?

**Problem 5**: How many ordered pairs $(x, y)$ of positive integers satisfy $2x + 5y = 100$?

**Problem 6**: Find the area of the circle that circumscribes a right triangle whose legs are of lengths 6 cm and 10 cm.

**Problem 7**: How many real roots does the equation $\log_{(x^2-3x)^3} 4 = \frac{2}{3}$ have?

**Problem 8**: Find the largest integer value of $n$ such that $1 \times 3 \times 5 \times 35$ is divisible by $3^n$.

**Problem 9**: Let $x$ be the solution of the equation

$$\frac{x+1}{1}+\frac{x+2}{2}+\frac{x+3}{3}+\cdots+\frac{x+2007}{2007}+\frac{x+2008}{2008}=2008.$$

Which of the following is true?

$(a) x > 1$ $(b) x = 1$ $(c) 0 < x < 1$ $(d) x \leq 0$

**Problem 10**: Let $0 < x < 1$. Which of the following has the largest value?

$(a) x^3$ $(b) x^2 + x$ $(c) x^2 + x^3$ $(d) x^4$

**Problem 11**: Find the sum of all the 4-digit positive numbers with no zero digit.

**Problem 12**: Find the number of real roots of the equation $x^5 - x^4 + x^3 - 4x^2 - 12x = 0$.

**Problem 13**: The roots of the quadratic equation $x^2 - 51x + k = 0$ differ by 75, where $k$ is a real number. Determine the sum of the squares of the roots.

**Problem 14**: Marco plans to give (not necessarily even) his eight marbles to his four friends. If each of his friends receives at least one marble, in how many ways can he apportion his marbles?

**Problem 15**: How many triangles (up to congruence) with perimeter 16 cm and whose lengths of its sides are integers?

**Problem 16**: The positive integer $n$, when divided by 3, 4, 5, 6, and 7, leaves remainders of 2, 3, 4, 5 and 6, respectively. Find the least possible value of $n$.

**Problem 17**: Find all positive numbers $x$ that satisfy

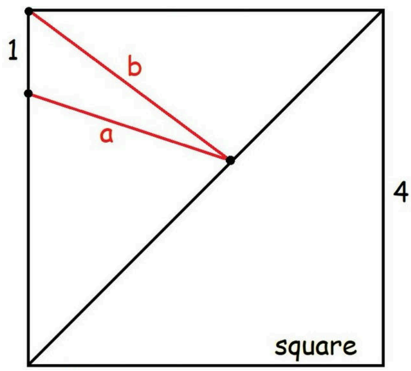
$$(2+\log_{10} x)^3 + (\log_{10} x - 1)^3 = (1+\log_{10} x^2)^3.$$

**Problem 18**: There are two times between 5 A.M. and 6 A.M. when the hands of an accurate clock are perpendicular. Exactly how many minutes must elapse between these two times?

**Problem 19**: Quadrialteral $ABCD$ has right angles at $B$ and $D$. If $ABCD$ is kite shaped with $AB = AD = 20$ and $BC = BD = 15$, find the length of a radius of the circle inscribed in $ABCD$.

**Problem 20**: Find the coordinates of that point on the circle with equation $(x-6)^2+(y-5)^2=25$ that is nearest to the point $(-2, 11)$.

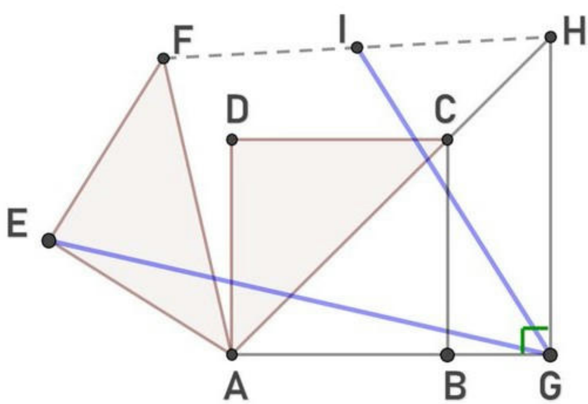
**Problem 21**: Find the minimum length of $a+b$ in the figure below.




**Problem 22**: If $a<b$, find the ordered pair of positive integers $(a, b)$ that satisfies

$$\sqrt{10+\sqrt{84}}=\sqrt{a}+\sqrt{b}.$$

**Problem 23**: $ABCD$ is a square. Rotate $ADC$ to $AEF$. $H$ is on ray $AC$ and $G$ is on ray $AB$. $\angle AGH=90^\circ$. $I$ is the middle point of $FH$. Find the length relationship between the two blue line segments.



**Problem 24**: If

$$x + y = 3xy$$
$$x^2 + y^2 = \frac{1}{3}$$

Find $\frac{x}{y}$.

**Problem 25**: Solve for $x$,

$$\left(4 + \sqrt{15}\right)^x + \left(4 - \sqrt{15}\right)^x = 62$$

**Problem 26**: Solve for $x$,

$$2(2^x - 1)x^2 + \left(2^{x^2} - 2\right)x = 2^{x+1} - 2$$

**Problem 27**: Solve in $\mathbb{R}$,

$$2^{x^2-3x} + 2^{x-x^2} = 2^{1-x}$$

**Problem 28**:

$$1 + \frac{3}{\phi} + \frac{5}{\phi^2} + \frac{7}{\phi^3} + ... = \phi^5$$

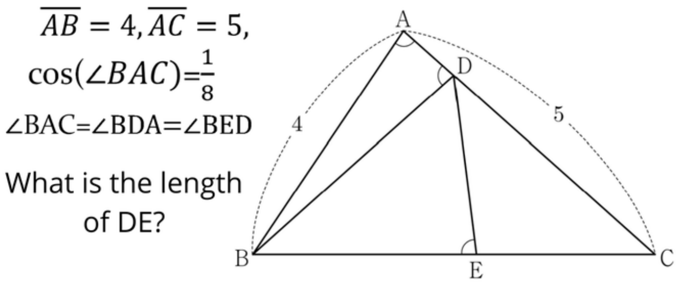
**Problem 29**: $p, q$ : roots of $x^2 - x - 1 = 0, p > q$

$$a_n = \frac{p^n - q^n}{p - q}, n \geq 1$$

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}, n \geq 2$

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} =?$$

**Problem 30**:



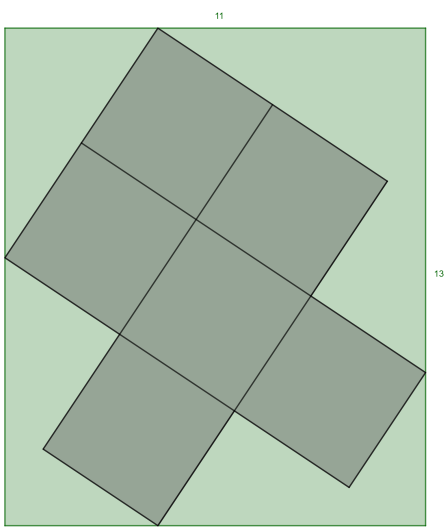

**Problem 31**:

$$\frac{1}{\sin\theta} - \frac{1}{\cos\theta} = \frac{\sqrt{13}}{6}$$

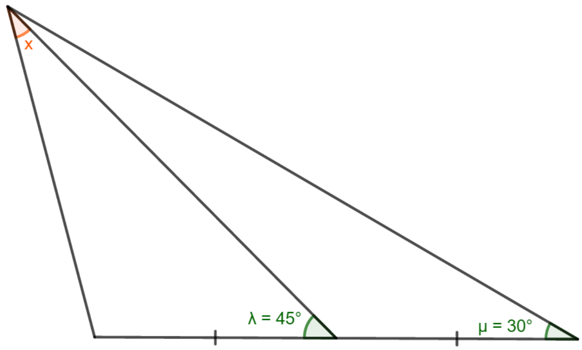
Find

$$\frac{1}{\tan\theta} - \tan\theta.$$

**Problem 32**: All small squares are of equal size and the big rectangle is $11 \times 13$. Find the area of the green region in the figure below.




**Problem 33**: Find angle $x$ in the figure below.




**Problem 34**: Find all real values of $x$ that satisfy $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$.

**Problem 35**: If $A$ and $B$ are digits and the base ten numeral $30AB5$ can be expressed as the product $225n$, find all positive integral values of $n$.

**Problem 36**: The equation of the line $m$ is $3x + 4y = 12$ in rectangular coordinate system. If the line whose equation is $3x + 4y = k$ is 2 units from $m$, find the two possible values of $k$.

**Problem 37**: Find all real values of $x$ that satisfy $|x| + 3 - |x + 3| = 6$.

**Problem 38**: Find all positive real values of $x$ that satisfy

$$\frac{1}{x+\sqrt{x}} + \frac{1}{x-\sqrt{x}} \leq 1.$$

**Problem 39**: The length of $\overline{AB}$ is 12, and $X$ is a variable point on $\overline{AB}$. Squares $AXCD$ and $XBEF$ are drawn in a plane on the same side of $\overline{AB}$. The centers of these squares are $Y$ and $Z$, respectively, and $W$ is the midpoint of $\overline{YZ}$. As $X$ moves from $A$ to $B$, find the length of the path traced out by point $W$.

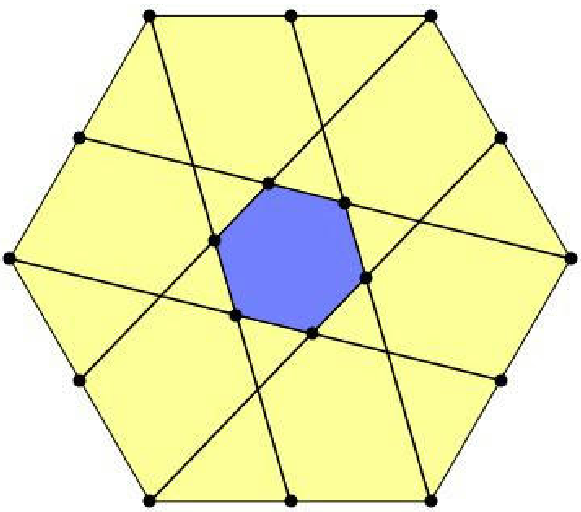
**Problem 40**: If $a$, $b$ and $c$ are different numbers and if $a^3 + 3a + 14 = 0$, $b^3 + 3b + 14 = 0$, and $c^3 + 3c + 14 = 0$, find the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

**Problem 41**: The distance between the circum centre and the in centre of a right angled triangle is 6 units. If both the circum radius and the inradius are positive integers, what is the area of the triangle?

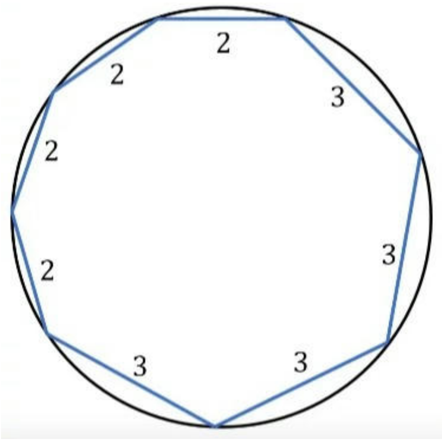
**Problem 42**: Prove that no number in the sequence $11, 111, 1111, 11111, \ldots$ is the square of an integer.

**Problem 43**: What is the remainder when $23^{23}$ is divided by 53?

**Problem 44**: Take a regular hexagon and connect each vertex to the midpoint of the opposite side, choosing the closest counter clockwise of the two opposite sides. This defines a central regular hexagon with the same centre as the starting hexagon. What proportion of the total area does the central hexagon occupy?

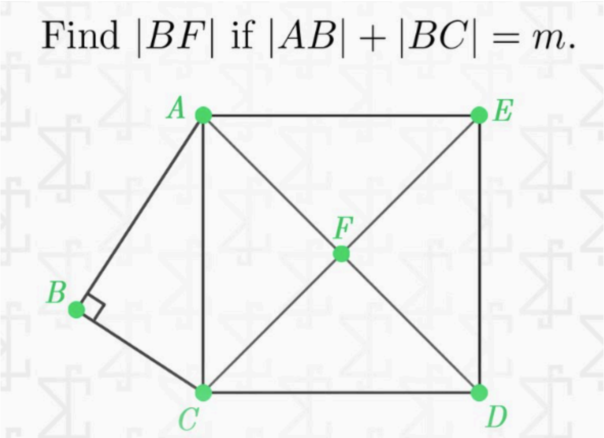


**Problem 45**: An octagon with the given side lengths is inscribed in a circle. Find the area of the octagon?




**Problem 46**: Consider a unit square. Four quarter circles of radius 1 are drawn with each of the vertices as center. What is the area of the region that is common to all the four quarter circles.

**Problem 47**:

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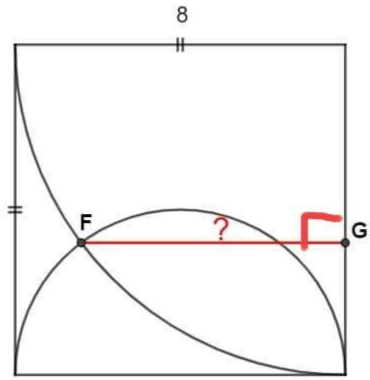
**Problem 48**: Find the length of $FG$ in the figure below:




**PROBLEM 49**: If

$$a_n = \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}1},$$

find $a_1 + a + 2 + ... + a_{99}$.

**PROBLEM 50**: Find the sum of all real roots of the equation:

$$\left(2\sqrt[5]{(x+1)} - 1\right)^4 + \left(2\sqrt[5]{(x+1)} - 3\right)^4 = 16.$$

**PROBLEM 51**: Assume that the quadratic equations

$$ax^2 + bx + c = 0$$

and

$$dx^2 + ex + f = 0$$

have exactly one solution $\alpha$ in common, then $\alpha$ =?

**PROBLEM 52**: In the Cartesian plane, what is the length of the shortest path from $(0,0)$ to $(8,6)$ that does not go inside the circle $(x-4)^2 + (y-3)^2 = \frac{75}{4}$?

**PROBLEM 53**: Consider a circle with center $O$. Let $PT$ be the tangent to the circle at $T$ and $R$ be the point diametrically opposite to $T$. If $PT$ = 72 cm, $PA$ = 54 cm and $TQ$ = 54 cm, then the radius of the circle is ?

**Problem 54**: Assume that $x, y \in \mathbb{R}$ satisfy the following system of equations given by

$$2x - 3y = \frac{x^2 + y^2}{5},$$
$$3x + 2y = \frac{8}{3}\sqrt{x^2 + y^2 + 4}.$$

If $x^2 + y^2 \leq 21$, determine $x + y$.

**Problem 55**: Find the sum of the numbers in the 12th row.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 3 & & \\ & 4 & & 5 & & 6 & \\ 7 & & 8 & & 9 & & 10 \\ \hline \end{array}$$

.

**Problem 56**: Determine the largest real number $z$ such that

$$x + y + z = 5$$
$$xy + yz + xz = 3$$

and $x, y$ are also real.

**Problem 57**: Find integer solutions to,

$$m^3 + n^3 + 99mn = 33^3$$

**Problem 58**: Each of the numbers $a_1, a_2, ..., a_n$ is 1 or $-1$, and we have $S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + ... + a_na_1a_2a_3 = 0$. Prove that $n$ is divisible by 4.

**Problem 59**: A $2 \times 3$ rectangle has vertices at $(0, 0)$, $(2, 0)$, $(0, 3)$, and $(2, 3)$. It rotates $90^{\circ}$ clockwise about the point $(2, 0)$. It then rotates $90\circ$ clockwise about the point $(5, 0)$, then $90^{\circ}$ clockwise about the point $(7, 0)$, and finally, $90^{\circ}$ clockwise about the point $(10, 0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1, 1)$.

**Problem 60**: A function $f$ is defined over the set of all positive integers and satisfies

$$f(1) = 1996$$

and

$$f(1) + f(2) + ... + f(n) = n^2 f(n)$$

for all $n > 1$. Calculate the exact value of $f(1996)$.

**Problem 61**:

For positive integers $n$, the sequence $a_1, a_2, a_3, ..., a_n, ...$ is defined by

$$a_1 = 1; a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + a_3 + ... + a_{n-1}), n > 1$$

Determine the value of $a_{1997}$.

**Problem 62**: Let $r$ be the number of real solutions and $l$ be the largest real solution of the equation
$|x^2 - 9| + |x - 4| + |x + 16| - 27 = 0$. Then find $l + r$.

**Problem 63**: British Maths Olympiad 2010 Round 1 Problem 1

One number is removed from the set of integers from 1 to $n$. The average of the remaining numbers is $40\frac{3}{4}$. Which integer was removed?

**Problem 64**: Find maximum and minimum of $a$,

$$a + b + c + d + e = 8$$
$$a^2 + b^2 + c^2 + d^2 + e^2 = 16$$

**Problem 65**: British Maths Olympiad 2000 Round 1 Problem 2

Show that, for every positive integer $n$,

$$121^n - 25^n + 1900^n - (-4)^n$$

is divisible by 2000.

**Problem 66**: Given that $x, y, z$ are positive real numbers satisfying $xyz = 32$, find the minimum value of

$$x^2 + 4xy + 4y^2 + 2z^2$$

**Problem 67**: Show that there are no integers $a, b, c$ for which $a^2 + b^2 - 8c = 6$.

**Problem 68**:

$$ax + by = 7$$
$$ax^2 + by^2 = 49$$
$$ax^3 + by^3 = 133$$
$$ax^4 + by^4 = 406$$

Find the value of $2014(x + y - xy) - 100(a + b)$.

**Problem 69**: Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

**Problem 70**:

Determine all pairs $(m, n)$ of positive integers which satisfy the equation

$$n^2 - 6n = m^2 + m - 10$$

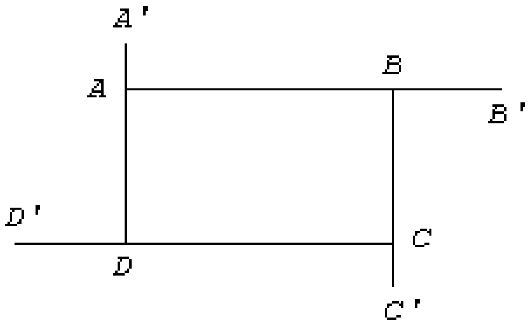
**Problem 71**: Let $x$ and $y$ be positive real numbers such that $x + y = 1$. Show that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9$$

**Problem 72**: The nine numbers from 1 to 9 are written on individual tiles. Using the nine tiles, form three three-digit numbers so that the sum of these three numbers is as large as possible. In how many different ways can the tiles be arranged?

**Problem 73**: A list of multiples of 9, starting with 9, is written in blue pencil. Next to each blue number, the sum of its digits is written in red pencil. Which appears first in the red list, the number 45 or a sequence of at least five 36s?

**Problem 74**: Let $BCD$ be a rectangle and $A'$, $B'$, $C'$, and $D'$ on the extensions of its sides such that: $AA' = k \cdot AD$; $BB' = k \cdot AB$; $CC' = k \cdot BC$; $DD' = k \cdot CD$. Find $k$ so that the area of quadrilateral $A'B'C'D'$ is 25 times the area of rectangle $ABCD$.

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**Problem 75**: Write a power of 2 at each vertex of a square. Then, on each side and each diagonal, write the product of the numbers assigned to its endpoints, such that the sum of the 10 numbers written is 3505.

**Problem 76**: Find all integer values of $x$ that satisfy:

$$2^x \cdot (4 - x) = 2x + 4.$$

**Problem 77**: With the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$, three numbers $A, B, C$ are formed, each with three different digits, using all nine digits. Can it be achieved that none of them are multiples of 3 ?

**Problem 78**: In a 50-meter race, if Daniel gives Gerardo a 4-meter head start, meaning Gerardo runs 46 meters, they finish together. In a 200-meter race, if Gerardo gives Marcelo a 15-meter head start, they finish together. How many meters head start should Daniel give Marcelo to finish together in a 1000-meter race? All three athletes run at constant speeds.

**Problem 79**: The family tree of a family begins with the marriage of Eduardo and Cecilia, who have three children: Orlando, Luis and Manuel. Of these three children, Orlando and Luis marry and Manuel remains single. For each of the following marriages the same situation is repeated (they have three children of which two get married and one remains single). Determine the number of people included in the family tree up to the tenth generation (include all spouses). Orlando, Luis and Manuel are from the first generation.

**Problem 80**: Find the number of elements in the set $\{(a, b) \in \mathbb{N} : 2 \leq a, b \leq 2023, \log_a(b) + 6\log_b(a) = 5\}$.

**Problem 81**: Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}.$$

Find the smallest possible value of β.

**Problem 82**: Let $X$ be the set of all even positive integers $n$ such that the measure of the angle of some regular polygon is $n$ degrees. Find the number of elements in $X$.

**Problem 83**: Given a $2 \times 2$ tile and seven dominoes ($2 \times 1$ tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a $2 \times 7$ rectangle using some of these tiles.

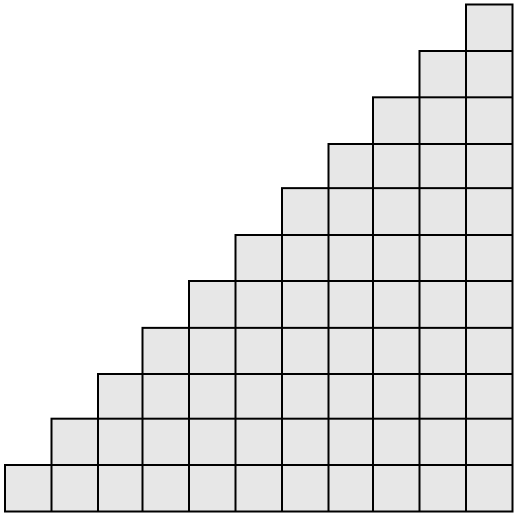
**Problem 84**: Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where $a, b, c$ are integers and $c$ is odd. Let $p_i$ be the value of $P(x)@x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

**Problem 85**: Let $x, y$ and $z$ be complex numbers such that $x + y + z = 2$, $x^2 + y^2 + z^2 = 3$ and $xyz = 4$. Then evaluate

$$\frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}.$$

**Problem 86**: How many trailing zeros does 365! have?

**Problem 87**: How many rectangles are in the 11×11 configuration illustrated?



A square counts as a rectangle. For example, in the corresponding $2 \times 2$ case, there are 5 rectangles.

**Problem 88**: Evaluate

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$$

where $n \in \mathbb{N}$.

**Problem 89**: Evaluate

$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \cdots + \frac{n}{(n+1)!},$$

where $n \in \mathbb{N}$.

**Problem 90**: Prove that for each $n \in \mathbb{N}$,

$$(n+1)(n+2)\cdots(2n)$$

is divisible by $2^n$.

**Problem 91**: Find the number of common positive divisors of $10^{40}$ and $20^{30}$.

**Problem 92**: Show that for any $n \in \mathbb{N}$, the number of positive divisors of $n^2$ is always odd.

**Problem 93**: There are two sets of parallel lines with $p$ and $q$ lines. Find the number of parallelograms formed by the line.

**Problem 94**: Show that among any 5 points in an equilateral triangle of unit side length, there are 2 whose distance is at most $\frac{1}{2}$ units apart.

**Problem 95**: Given any set $C$ of $n + 1$ distinct points ($n \in \mathbb{N}$) on the circumference of a unit circle, show that there exist $a, b \in C, a \neq b$, such that the distance between them does not exceed $2 \sin \frac{\pi}{n}$.

**Problem 96**: Given any set $S$ of 9 points within a unit square, show that there always exist 3 distinct points in $S$ such that the area of the triangle formed by these 3 points is less than or equal to $\frac{1}{8}$.

**Problem 97**: Show that given any set of 5 numbers, there are 3 numbers in the set whose sum is divisible by 3.

**Problem 98**: Let $A$ be a set of $n + 1$ elements, where $n \in \mathbb{N}$. Show that there exist $a, b \in A$ with $a \neq b$ such that $n \mid (a - b)$.

**Problem 99**: Let $A = \{a_1, a_2, ..., a_{2k+1}\}$, where $k \geq 1$, be a set of $2k + 1$ positive integers. Show that for any permutation $a_{i_1}, a_{i_2}, ..., a_{i_{2k+1}}$ of $A$, the product

$$\prod_{j=1}^{2k+1} \left(a_{i_j} - a_j\right)$$

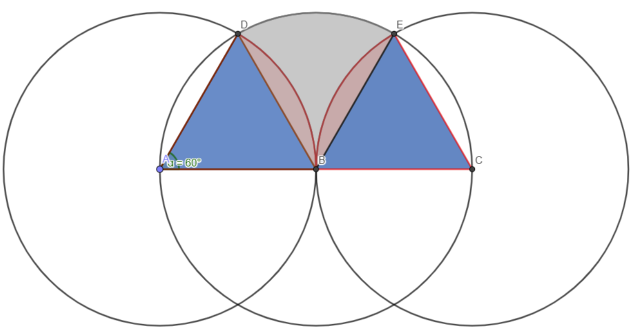
is always even.

**Problem 100**: Let $A \subseteq \{1, 2, ..., 2n\}$ such that $\mid A \mid = n + 1$ , when $n \in \mathbb{N}$. Show that there exist $a, b \in A$, with $a \neq b$ such that $a \mid b$.

# Solutions

**Solution 1**: For the area of triangle $OAB$ to be atleast $\frac{1}{4}$, the height of the triangle should be greater than or equal to $\frac{1}{2}$.The point $B$ should be on the arcs of circle between $\frac{1}{2} \leq x \leq 1 \vee -1 \leq x \leq -\frac{1}{2}$. The two arcs subtend an angle of $240^{\circ}$ therefore the required probability is $\boldsymbol{\frac{2}{3}}$.

**Solution 2**: We have $AD = AB = BD = BE = EC = BC = 4cm$ as they are all radii. Therefore the triangles $ABD$ and $BCE$ are equilateral. The anlge $DBE$ is $180^{\circ} - 2 \times 60^{\circ} = 60^{\circ}$. Let the area of the grey region be $G$, the area of the sector $EDB$ be $S$, the area of the red region be $R$ and the area of the blue region be $B$. We have $G = S - 2R$. The area of the sector $ABD$ is also $S$ as the $\angle DAB = 60^{\circ}$. Therefore, we have $S = B + R$. Combining the above two equations, we have $G = S - 2(S - B) = 2B - S$. The reason for rewriting $G$ in terms of $S$ and $B$ is that we have formulae for calcualting the areas of sectors and equilateral triangles.
$B = \frac{\sqrt{3}}{4} \times 4^2 cm^2, S = \frac{\pi}{6} \times 4^2 cm^2, G = \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) 4^2 cm^2$. The total grey area in the original figure is $2G = \boldsymbol{16\left(\sqrt{3} - \frac{\pi}{3}\right) cm^2}$.




**Solution 3**: The line joining $O(-1, 3)$, the center of the circle and point $A(6, 8)$ is perpendicular to the tangent at $A$. The slope of the line joining $OA$ is $\frac{8-3}{6-(-1)} = \frac{5}{7}$. Therefore the slope of the tangent at $A$ is $-\frac{7}{5}$. The tangent at $A$ has a slope $-\frac{7}{5}$ and passes through the point $A(6, 8)$ so it has the equation $y - 8 = \left(-\frac{7}{5}\right)(x - 6) \Rightarrow \boldsymbol{7x + 5y - 82 = 0}$.

**SOLUTION 4**: We see that

$$AP^2 = AM^2 + MP^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = \mathbf{\frac{5}{8}}.$$

**SOLUTION 5**: We have,

$$x = \frac{5(20-y)}{2} \Rightarrow y \in \{2, 4, ..., 18\}.$$

For each value of $x$, we have one value of $y$, therefore the number of ordered pairs $(x, y)$ of positive integers which satisfy the above equation is **9**.

**SOLUTION 6**: The circumcenter is the midpoint of the hypotenuse so the radius is half the length of the hypotenuse. Therefore the area of the circle is $\pi\frac{\sqrt{6^2+10^2}}{2} = \boldsymbol{68\pi cm^2}$.

**SOLUTION 7**: We have
$x^2 - 3x > 0 \wedge x^2 - 3x \neq 1 \wedge (x^2 - 3x) = 2$. Therefore the equation has **two** real roots.

**SOLUTION 8**: The largest power of 3 in 35! is given by
$\lfloor\frac{35}{3}\rfloor + \lfloor\frac{35}{3^2}\rfloor + \lfloor\frac{35}{3^3}\rfloor = 15$. The largest power of 3 in
$2 \times 4 \times 6 \times ... \times 34$ is given by $\lfloor\frac{17}{3}\rfloor + \lfloor\frac{17}{3^2}\rfloor = 6$. Therefore the largest value of $n$ is $15 - 6 = \mathbf{9}$.

**SOLUTION 9**: Simplifying the LHS, we get

$$x\left(1 + \frac{1}{2} + ... + \frac{1}{2008}\right) = 0.$$

Therefore we see that $(\boldsymbol{d})$ is true.

**SOLUTION 10**: When $0 < x < 1$, we have $x^n < x$ when
$n > 1 \wedge n \in \mathbb{N}$. Therefore, $\boldsymbol{x^2 + x}$ has the largest value.

**SOLUTION 11**: The total number of 4-digit numbers satisfying the criteria is $9^4$.In each place each digit appears $9^3$ times.Therefore the sum of all 4-digit positive numbers with no zero digit is

$$9^3 \cdot \left(\sum_{k=1}^{9} k\right) \cdot (10^3 + 10^2 + 10 + 1) = \mathbf{36446355}.$$

**Solution 12**: We have,

$$\begin{aligned} x(x^4 - x^3 + x^2 - 4x - 12) &= 0 \\ \Rightarrow x((x^2-3)(x^2+4) - x(x^2+4)) &= 0 \\ \Rightarrow x(x^2+4)(x^2-x-3) &= 0. \end{aligned}$$

Therefore the number of real roots of the equation is **three**.

**Solution 13**: The sum of the squares of the roots is $75^2 + 2k = 51^2 - 2k \Rightarrow k = -126 \cdot 6$. Therefore the sum of the squares of the roots is $75^2 - 12 \cdot 126 = \mathbf{4113}$.

**Solution 14**: The number of ways of apportioning the marbles is the number of positive solutions of the equation $a + b + c + d = 8$ or the number of non-negative solutions of the equation $a + b + c + d = 4$ which is $\binom{\mathbf{4+4-1}}{\mathbf{4}} = \mathbf{35}$.

**Solution 15**: If $x, y$ and $z$ are the sides of the triangle, we have $x + y + z = 16, x + y > z$,$x + z > y$ and $y + z > x$. Hence, $8 < x, 8 < y$ and $8 < z$. The triangles can only have lengths $(2, 7, 7)$, $(3, 6, 7)$, $(4, 6, 6, )$, $(4, 5, 7)$, $(6, 5, 5)$, so there are **five** triangles.

**Solution 16**: $n = 3a + 2 = 4b + 3 = 5c + 4 = 6d + 5 = 7e + 6$. Then $n + 1$ is divisible by $3, 4, 5, 6, 7$. The smallest such positive $n + 1$ is LCM of $3, 4, 5, 6, 7$ is 420. Therefore, the smallest such positive $n$ is 419.

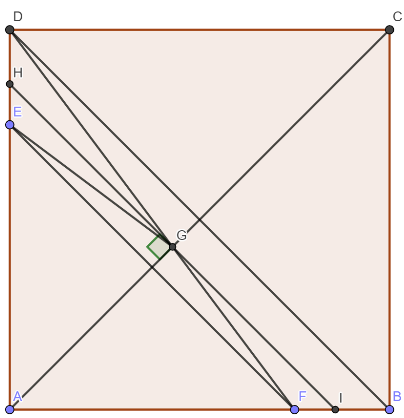
**Solution 17**: The equation is of the form $a^3 + b^3 = (a + b)^3 \Rightarrow ab(a + b) = 0$. Therefore $x = 10$ or $x = \frac{1}{100}$ or $x = \frac{1}{\sqrt{10}}$.

**Solution 18**: Between the two times, if the hour hand covered $x^\circ$, the minute hand should have covered $(180 + x^\circ)$. The required time is $\frac{180}{5.5} = 32\frac{8}{11}$ as the relative angular speed of the minute hand w.r.t. the hour hand is $\frac{5.5^\circ}{min}$.

**Solution 19**: The area of the kite $ABCD$ is $2 \times \frac{1}{2} \times 20 \times 15 = 300$. The perimeter of the kite is 70. If $r$ is the radius of the inscribed circle, we have $\frac{1}{2} \times 70 \times r = 300 \Rightarrow r = \frac{60}{7}$.

**Solution 20**: The point on the circle nearest to the point $A(-2, 11)$ is on the line joining center of the circle to $A$. The distance between $A$ and $O(6, 5)$ the center of the circle is 10 and the radius of the circle is 5. Therefore the required point is the midpoint of $\overline{OA}$ which is $(2, 8)$.

**Solution 21**: For $a + b$ to be minimum, the red line segements should be part of a path a ray of light traverses starting at $D$ and exiting at $E$ after getting reflected at the diagonal $AC$ as in the figure below. $DF$ is the path a ray of light would have taken without reflection. $GH$ is perpendicular to $AC$ and parallel to $DB$ and $EF$. It is easy to see that $\angle DGH = \angle HGE$ and $GE = GF$. We have $DA = 4$, $AF = AB - BF = AB - DE = 3$. Therefore $\min(a + b) = DG + GE = DF = \sqrt{DA^2 + AF^2} = \mathbf{5}$.




.

**Solution 22**: It is easy to see that $\sqrt{10 + \sqrt{84}} = \sqrt{10 + 2\sqrt{21}} = \sqrt{3} + \sqrt{7}$.

**Solution 23**: Let the coordinates of $A$ be $(0, 0)$, the coordinates of $B, C$ and $D$ be $(1, 0)$, $(1, 1)$ and $(0, 1)$ respectively. Let $\triangle\ ADC$ be rotated by $\theta^\circ$. Let $\overrightarrow{AG} = s\boldsymbol{i}$. We have the following:

$$\overrightarrow{AH} = s\boldsymbol{i} + s\boldsymbol{j}.$$

$$\overrightarrow{AE} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} \text{ as } \overrightarrow{AE} \text{ is } \overrightarrow{AD} \text{ rotated by } \theta^\circ .$$

$$\overrightarrow{GE} = \overrightarrow{AE} - \overrightarrow{AG} = -(s + \sin\theta)\boldsymbol{i} + \cos\theta\boldsymbol{j}$$

$$\Longrightarrow |\ \overrightarrow{GE}\ | = \sqrt{1 + s^2 + 2s\sin\theta}.$$

$$\overrightarrow{AF} = (\cos\theta - \sin\theta)\boldsymbol{i} + (\sin\theta + \cos\theta)\boldsymbol{j}$$

$$\text{as } \overrightarrow{AF} \text{ is } \overrightarrow{AC} \text{ rotated by } \theta^\circ .$$

$$\overrightarrow{AI} = \frac{1}{2}\left(\overrightarrow{AF} + \overrightarrow{AH}\right)$$

$$= \frac{1}{2}((s + \cos\theta - \sin\theta)\boldsymbol{i} + (s + \sin\theta + \cos\theta)\boldsymbol{j}).$$

$$\overrightarrow{GI} = \overrightarrow{AI} - \overrightarrow{AG} = \frac{1}{2}((\cos\theta - \sin\theta - s)\boldsymbol{i} + (s + \sin\theta + \cos\theta)\boldsymbol{j}).$$

$$|\ \overrightarrow{GI}\ | = \frac{1}{\sqrt{2}}\sqrt{1 + s^2 + 2s\sin\theta}.$$

Therefore, $|\ \overrightarrow{\boldsymbol{GI}}\ | = \boldsymbol{\frac{1}{\sqrt{2}}}\ |\ \overrightarrow{\boldsymbol{GE}}\ |$.

**Solution 24**: We have,

$$x^2 + y^2 + 2xy = 9x^2y^2$$

$$\Longrightarrow \frac{1}{3} + 2xy = 9x^2y^2$$

$$\Longrightarrow xy = \frac{1}{3} \vee xy = -\frac{1}{9}$$

When $xy = \frac{1}{3}$ and $x + y = 1$,

$$\frac{x}{y} = \frac{-\sqrt{3} + 3i}{\sqrt{3} + 3i} \vee \frac{x}{y} = \frac{\sqrt{3} + 3i}{-\sqrt{3} + 3i}$$

When $xy = -\frac{1}{9}$ and $x + y = -\frac{1}{3}$,

$$\frac{x}{y} = \frac{1}{2}\left(-3 - \sqrt{5}\right) \vee \frac{x}{y} = \frac{1}{2}\left(\sqrt{5} - 3\right)$$

**Solution 25**: Let $\left(4 + \sqrt{15}\right)^x = y$. We have,

$$y + \frac{1}{y} = 62$$
$$\implies y = 31 + 8\sqrt{15} \vee y = 31 - 8\sqrt{15}$$

Therefore, $x = 2$ or $x = -2$.

**Solution 26**: Rearranging the above equation, we get

$$(2^x - 1)(x^2 - 1) = -x\left(2^{x^2-1} - 1\right)$$

It is easy to see that LHS is 0 for $x = 0$ and $x \pm 1$. For these values of $x$, the RHS is 0 as well. For $x > 1$ or $x < -1$, the LHS is positive but RHS is negative, therefore the equation cannot have any solutions for $x > 1$ or $x < -1$. For $0 < x < 1$, the LHS is negative but RHS is positive, therefore the equation cannot have any solutions for $0 < x < 1$. For $-1 < x < 0$, the LHS is positive but RHS is negative, therefore the equation cannot have any solutions for $-1 < x < 0$. Therefore $x = 0$ and $x \pm 1$ are the only solutions.

**Solution 27**:

$$2^{(x-1)^2-(x-1)-2} + 2^{-(x-1)^2-(x-1)} = 2^{-(x-1)}$$
$$\implies 2^{(x-1)^2-2} + 2^{-(x-1)^2} = 1$$
$$\implies 2^{(x-1)^2} = 2$$
$$\implies x = 0 \vee x = 2$$

**Solution 28**:

$$S = 1 + \frac{3}{\phi} + \frac{5}{\phi^2} + \frac{7}{\phi^3} + \ldots$$
$$\implies \phi S = \phi + 3 + \frac{5}{\phi} + \frac{7}{\phi^2} + \ldots$$
$$\implies S(\phi - 1) = \phi + 2 + \frac{2}{\phi} + \frac{2}{\phi^2} + \ldots$$
$$\implies S(\phi - 1) = \phi + \frac{2\phi}{\phi - 1} = \phi\frac{\phi + 1}{\phi - 1}$$
$$\implies S = \frac{\phi^3}{(\phi - 1)^2} = \phi^5$$

**Solution 29**: We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{b_n}{10^n} &= \frac{1}{10} + \sum_{n=2}^{\infty} \frac{a_{n-1}}{10^n} + \frac{a_{n+1}}{10^n} \\
&= \frac{1}{10} + \frac{1}{p-q}\left(\frac{1}{10}\sum_{n=1}^{\infty}\frac{p^n-q^n}{10^n} + 10\sum_{n=2}^{\infty}\frac{p^{n+1}-q^{n+1}}{10^{n+1}}\right) \\
&= \frac{1}{10} + \frac{1}{p-q}\left(\frac{1}{10}\left(\frac{p}{10-p} - \frac{q}{10-q}\right) + \frac{1}{10}\left(\frac{p^3}{10-p} - \frac{q^3}{10-q}\right)\right) \\
&= \frac{1}{10} + \frac{1}{p-q}\left(\frac{p-q}{100-10p-10q+pq} + \frac{1}{10}\cdot\frac{10p^3 - pq(p^2-q^2) - 10q^3}{100-10p-10q+pq}\right) \\
&= \frac{1}{10} + \frac{1}{89} + \frac{10(p^2+q^2+pq) - pq(p+q)}{10\cdot 89} \\
&= \frac{1}{10} + \frac{1}{89} + \frac{21}{10\cdot 89} \\
&= \frac{12}{89}
\end{aligned}$$

**Solution 30**: We have $BD = 4$ as $\angle BAC = \angle BDA$.

Using Cosine rule for $\triangle\, ABC$, we have

$$|BC| = \sqrt{4^2 + 5^2 - 2\cdot 4\cdot 5\frac{1}{8}} = 6$$

Using Cosine rule for $\triangle\, BDC$,

$$|BC| = 6 = \sqrt{4^2 + |DC|^2 + 2\cdot 4\cdot |DC|\frac{1}{8}} \Longrightarrow |DC| = 4.$$

Let $F$ be the perpendicular from $D$ to $BC$.As $\triangle\, BDC$ is isoceles, $\angle BDF = (180^\circ - \angle BDA)/2$. Therefore,

$$|DE| = \frac{4\sin(\angle BDA/2)}{\sin\angle BED} = \frac{2}{\cos(\angle BDA/2)} = \frac{8}{3}$$

as $\cos\angle BDA = \frac{1}{8} = 2\cos^2(\angle BDA/2) - 1$.

**Solution 31**: Let

$$u = \frac{1}{\sin\theta}, v = \frac{1}{\cos\theta}$$

We have,

$$\begin{aligned} u - v &= \frac{\sqrt{13}}{6} \\ \frac{1}{u^2} + \frac{1}{v^2} &= 1 \\ \implies \frac{13}{36} + 2uv &= u^2v^2 \\ \implies uv = -\frac{1}{6} &\vee uv = \frac{13}{6} \end{aligned}$$

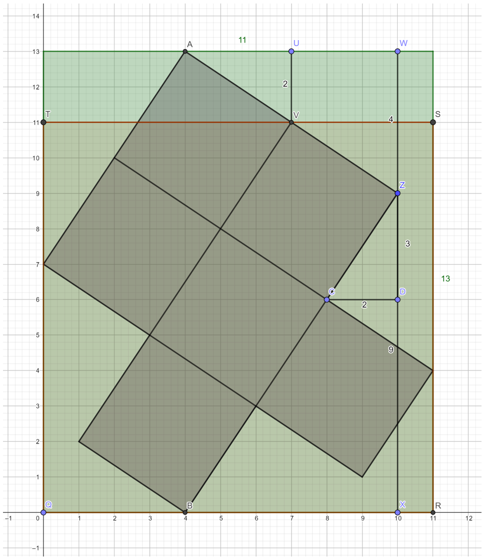
We now have,

$$u + v = \pm\sqrt{(u-v)^2 + 4uv} = \pm 5\frac{\sqrt{13}}{6} \text{ , when } uv = \frac{13}{6}.$$

Therefore,

$$\frac{1}{\tan\theta} - \tan\theta = \frac{u}{v} - \frac{v}{u} = \frac{(u+v)(u-v)}{uv} = \pm\frac{5}{6}.$$

**Solution 32**: The key insight is that the cross comprising of 5 squares is within the $11 \times 11$ square $QRST$.

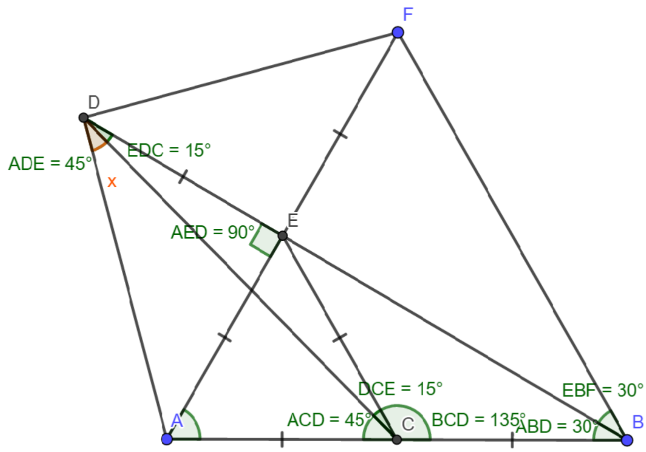



So we now have $UV = 2$.From the similar triangles $AUV$ and $AWZ$ it is easy to see that $WZ = 4$. Hence, $ZX = WX - WZ = 13 - 4 = 9$. From the similar triangles $ZCD$ and $ZBX$, we have $ZD = 3$.

Triangles $AUV$ and $ZDC$ are congruent, so the side of the small square $AV = ZC = \sqrt{2^2 + 3^2} = \sqrt{13}$.

Therefore, the area of the green region is $13 \cdot 11 - 13 \cdot 6 = 65$.

**Solution 33**: Reflecting $\triangle ADB$ about segment $DB$ we get the equilateral triangle $\triangle ABF$ because $AB = AF$ and $\angle BF = 60°$.




As $AB = BF$, we also have $CE = \frac{1}{2}BF = AC$ as $C$ and $E$ are the midpoints of $AF$ and $AB$.

Triangle $AEC$ is equilateral because we have $AC = CE$ and $\angle FAB = \angle EAC = 60^\circ$. We now have $AE = EC = AC$.

We also have
$\angle CDB = 180^\circ - \angle CBD - \angle BCD = 180^\circ - 30^\circ - (180^\circ - 45^\circ) = 15^\circ$,
$\angle ECD = \angle ECA - \angle ACD = \angle 60^\circ - 45^\circ = 15^\circ$. Therefore,
$DE = EC = EA$.

As $DE = EA$, $\angle ADE = \angle DAE = 45^\circ$ because $\angle AED = 90^\circ$.

Therefore $x = \angle ADE - \angle CDB = 45^\circ - 15^\circ = 30^\circ$.

**Solution 34**: We have
$(x-1)^4 - x^2 = 0 \Rightarrow (x^2 - 3x - 1)(x^2 - x + 1) = 0$. Solving the quadratic equations for real $x$ yields only two solutions $\frac{3\pm\sqrt{5}}{2}$.

**Solution 35**: As $225 = 25 \times 9$, $3A0B5$ should be divisible by 25 and 9. Therefore $B = 2$ or $B = 7$. When $B = 2$, $A = 8$ and when $B = 7$, $A = 3$. Therefore $n = 132$ or $n = 137$.

**Solution 36**: The distance from the origin to the line $m$ given by $3x + 4y = 12$ is $\frac{12}{5}$. Lines parallel to $m$ which are at a distance of 2 units from it would have a distance of $\frac{22}{5}$ or $\frac{2}{5}$ from the origin. The two possible values of $k$ are 22 and 5.

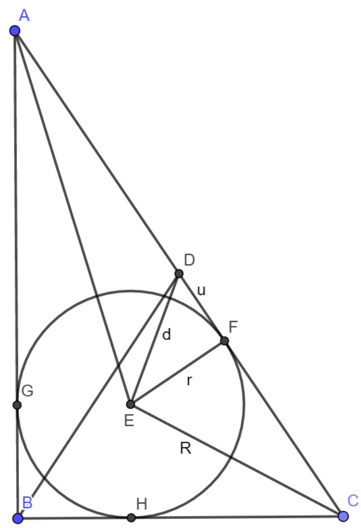
**Solution 37**: It is easy to see that there are no solutions for $-3 < x < 0$ or when $x \geq 0$. Therefore only when $x \leq -3$, the equation is satisfied.

**Solution 38**: We have $x > 0$, therefore $x + \sqrt{x} \neq 0$. For $x - \sqrt{x} \neq 0$, we need $x \neq 1$. The inequality reduces to $\frac{2}{x-1} \geq 1 \Rightarrow 2(x-1) \geq (x-1)^2 \Rightarrow (x-1)(x-3) \geq 0$. Therefore the set of values that satisfy the inequality is $0 < x < 1$ and $x \geq 3$.

**Solution 39**: Let the coordinates of A be $(0, 0)$ and B be $(0, 12)$. The coordinates of X are $(x, 0)$, Y are $\left(\frac{x}{2}, 0\right)$ and Z are $\left(\frac{12+x}{2}, \frac{12-x}{2}\right)$. The coordinates of W are $\left(3 + \frac{x}{2}, 3\right)$. When $0 \leq x \leq 12$, W traces the line segment between $(3, 3)$ and $(3, 9)$ which has a length of 6.

**Solution 40**: $a, b$ and $c$ can be considered roots of the equation $x^3 + 3x + 14 = 0$. From the equation we see that $abc = -14$ and $ab + bc + ca = 3$. Therefore, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = -\frac{3}{14}$

**Solution 41**: In the figure below,



$\Delta ABC$ is a right angled triangle with $\angle B = 90^{\circ}$. Let $r$ be the in radius and $R$ be the circum radius of the triangle. $D$ is the mid point of $AC$. Let $DF = u$.

As $D$ is the circumcentre, we have $AD = DC = BD = R$. We have $AF = AG = R + u$, $CF = CH = R - u$, $GB = BH = r$, hence $AB = AG + GB = R + u + r$ and
$BC = CH + HB = R - u + r$.

Using the **Apollonius theorem** in triangle $\Delta ABC$, we have

$$AB^2 + BC^2 = 2(BD^2 + CD^2)$$
$$\implies (R + u + r)^2 + (R - u + r)^2 = 4R^2$$
$$\implies u^2 + r^2 + 2rR = R^2$$

From the right angled triangle $\Delta DEF$, we also have $u^2 + r^2 = d^2$.

Eliminating $u$ from the above two equations, we get
$d^2 + 2rR = R^2$.

As $d = 8$, $r$ and $R$ are integers, from the equation
$R(R - 2r) = 64$, we get $R = 16$ and $r = 6$.

Area of the triangle
$\Delta ABC = r \cdot s = r\left(\frac{AB+BC+CA}{2}\right) = r(2R + r) = 6 \cdot 38 = 228$.

**Solution 42**: All terms in the above sequence are of the form $4k + 3$ and no square integer can have that form.

**Solution 43**: $23^2 = 529 \equiv -1 \bmod (53)$.
$23^{22} \equiv (-1)^{11} \bmod (53)$. Therefore,
$23^{23} \equiv -23 \bmod (53) \equiv 30 \bmod (53)$.

**Solution 44**: The inner hexagon is 1/13 of the starting hexagon. This can be derived in various ways, using geometry or trigonometry and algebra, but the discoverer, Rick Mabry, had the cute dissection solution shown in the diagram below.

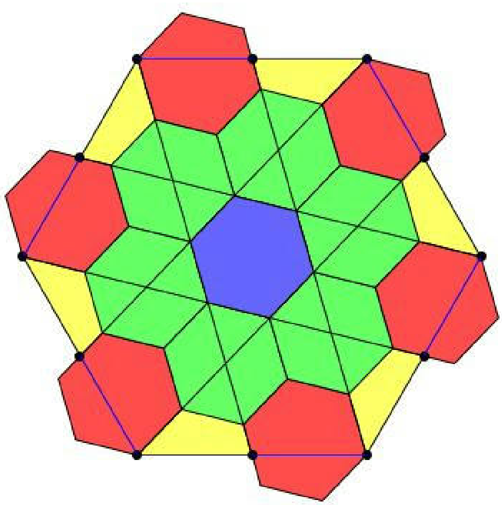


Figure 1:

**Solution 45**: The octagon can be transformed as follows while preserving the area

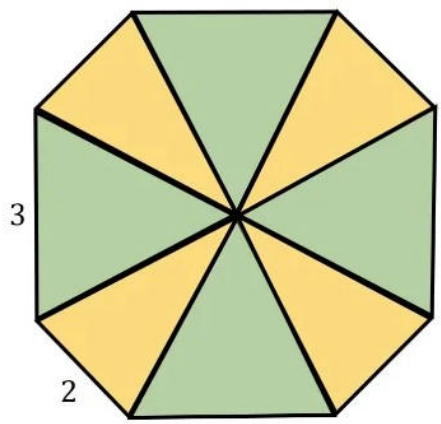



Figure 2:

The side of the square is $3 + 2\sqrt{2}$. The area of the octagon is $\left(3 + 2\sqrt{2}\right)^2 - 4 \cdot \frac{1}{2} \cdot \left(\sqrt{2}\right)^2 = 13 + 12\sqrt{2}$.

**Solution 46**: The common area to all the four quarter circles is the area bounded by the circular arcs $LN$, $NO$, $OP$ and $PL$ in the diagram below. From symmetry it is easy to see that the area is 8 times the area of the region $LEM$. The area of the region $LEM$ =

Area of the sector $ALM$ - Area of the $\Delta AEL$. $\Delta LAB$ is an equilateral triangle, therefore $\angle LAE = \angle LAB - \angle EAB = 60^\circ - 45^\circ = 15^\circ$. The area of the sector $ALM = \frac{\pi}{24}$. The area of $\Delta AEL = \frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{2}} \cdot \sin 15^\circ$. Hence the required area is given by $8\left(\frac{\pi}{24} - \frac{\sin 15^\circ}{2\sqrt{2}}\right) = \frac{\pi}{3} - \sqrt{3} + 1$.

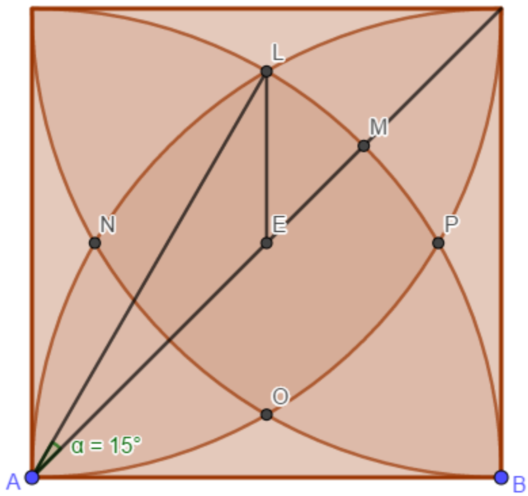
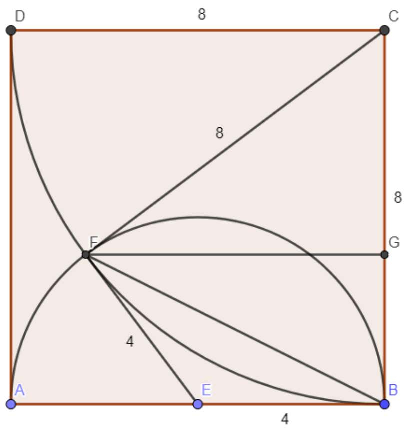



Figure 3:

**Solution 47**: $ABCF$ are the vertices of a concyclic quadrilateral. From **Ptolemy's Theorem**, we have

$$BF \cdot AC = AB \cdot CF + BC \cdot AF \Rightarrow BF = \frac{m}{\sqrt{2}}.$$

**Solution 48**:




We see that $\sin(\angle FCB) = \sin(2\angle ECB) = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{4}{5}$.

Area of $\Delta CFB = \frac{1}{2} \cdot 8 \cdot 8 \cdot \frac{4}{5} = \frac{1}{2} \cdot 8 \cdot FG \Rightarrow \boldsymbol{FG = \frac{32}{5}}$.

**Solution 49**: We have,

$$a_n = \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}.$$

Therefore

$$a_1 + a_2 + \ldots + a_{99} = \frac{9}{10}.$$

**Solution 50**: Let $2\sqrt[5]{x+1} - 2 = y$. We have

$$(y+1)^4 + (y-1)^4 = 16$$
$$\Rightarrow y^4 + 6y^2 - 7 = 0$$
$$\Rightarrow (y^2+7)(y^2-1) = 0.$$

The real roots of the above equation are $y = \pm 1$. When $y = -1, x = \frac{211}{32}$ and when $y = 1, x = -\frac{31}{32}$. Therefore the sum of real roots of the equation is $\boldsymbol{\frac{45}{8}}$.

**Solution 51**: We have,

$$a\alpha^2 + b\alpha + c = 0$$
$$d\alpha^2 + e\alpha + f = 0.$$

From the above equations, we have

$$\frac{b}{a}\alpha + \frac{c}{a} = \frac{e}{d}\alpha + \frac{f}{d} \Rightarrow \alpha = \boldsymbol{\frac{af - cd}{bd - ae}}.$$

**Solution 52**: The shortest path consists of a straight line from $O(0,0)$ to $A$, the arc $AB$ of the circle and the straight line from $B$ to $D(8,6)$. The length of the shortest path is $2 \cdot \frac{5}{2} + \frac{5\sqrt{3}}{2} \cdot \frac{2\pi}{3} = \mathbf{5} + \boldsymbol{\frac{5\pi}{\sqrt{3}}}$.

**Solution 53**: We have $PT^2 = PA \cdot (PA + AB)$. Hence $AB = 42$ cm. $\Delta TAQ \sim \Delta BRQ \Rightarrow \frac{AQ}{QR} = \frac{TQ}{QB}$. We also have $(PA + AQ)^2 = PT^2 + TQ^2 \Rightarrow AQ = 36$ cm and $QB = 6$ cm. Therefore we have $\frac{36}{2r-54} = \frac{54}{6} \Rightarrow r = \mathbf{29}$ cm.

**Solution 54**: Squaring and adding both the equations we have,

$$13(x^2 + y^2) = \frac{(x^2 + y^2)^2}{25} + \frac{64}{9}(x^2 + y^2 + 4).$$

Let $u = x^2 + y^2$, we have the quadratic $13u = \frac{u^2}{25} + \frac{64}{9}(u + 4) \Rightarrow u = 5$ as $u \leq 21$. We now have the following system of equations

$$2x - 3y = 1$$
$$3x + 2y = 8.$$

Therefore $x + y = \mathbf{3}$.

**Solution 55**: It is easy to see that there will be $n$ numbers in the $n^{th}$ row starting with $1 + \frac{n \cdot (n-1)}{2}$. Therefore the sum of the numbers in the $12^{th}$ row is $\frac{12}{2}(67 + (67 + 12 - 1)) = \mathbf{870}$.

**Solution 56**: As $x$ and $y$ are real, $(x+y)^2 \geq 4xy$ which implies

$$(5 - z)^2 \geq 4(3 - z(5 - z))$$
$$\Longrightarrow 3z^2 - 10z - 13 \leq 0$$
$$\Longrightarrow -1 \leq z \leq \frac{13}{3}$$

**Solution 57**:

$$m^3 + n^3 + 99mn = 33^3, m, n \in \mathbb{N}$$

$$\implies \left(\frac{m}{33}\right)^3 + \left(\frac{n}{33}\right)^3 + 3\left(\frac{m}{33}\right)\left(\frac{n}{33}\right) = 1$$

$$\implies (M+N)^3 - 3MN(M+N-1) = 1, M = \frac{m}{33}, N = \frac{m}{33}$$

$$\implies (M+N)^3 - 1 = 3MN(M+N-1)$$

$$\implies M+N-1 = 0 \vee M^2 + N^2 - MN + 1 + M + N = 0$$

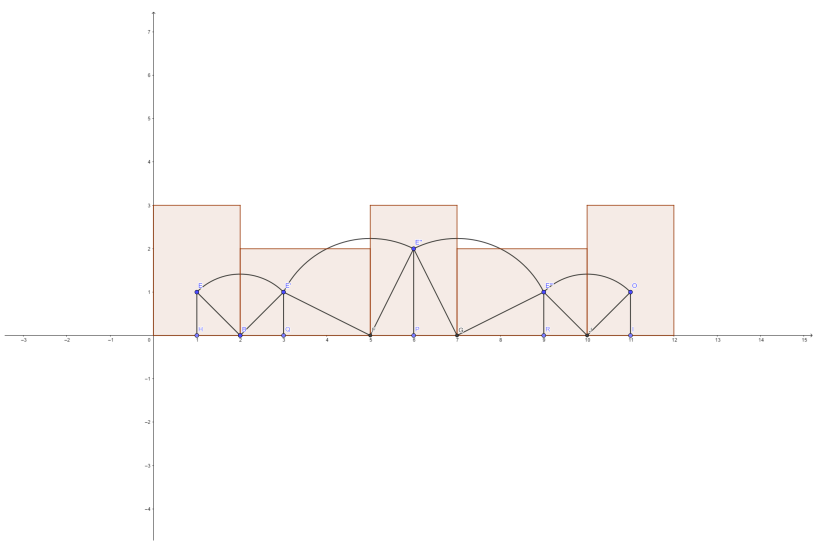
$$\implies M+N-1 = 0 \vee (M+1)^2 + (N+1)^2 - (M+1)(N+1) = 0$$

$$\implies M+N-1 = 0 \vee M = -1 \wedge N = -1$$

$$\implies m+n = 33 \vee m = -33 \wedge n = -33$$

**Solution 58**: It is easy to see that $n$ cannot be odd as to get a sum of zeros you need 1 and $-1$ to occur in pairs, so $n$ cannot be of the form $4k+1$ and $4k+3$. So $n$ can only be of the form $4k$ or $4k+2$. If $n = 4k+2$, you would have $2k+1$ terms in the sum $S$ which are $-1$ and $2k+1$ which are 1. The product of all the $4k+2$ terms will be $-1$ but the product of all the terms in $S$ which is $(a_1 ... a_n)^4$ is 1 which is a contradiction. Therefore, $n$ has to be of the form $4k$.

**Solution 59**: Here is the diagram showing how the Point $E(1,1)$ moves as the rectangle rotates



It is easy to see that the required area is

$$2\left(\frac{\pi}{4}((1^2+1^2)+(2^2+1^2))+2\left(\frac{1}{2}\cdot 1\cdot 1+\frac{1}{2}\cdot 1\cdot 2\right)\right)=3.5\pi+6.$$

**Solution 60**: We have

$$\begin{aligned} f(n+1) &= (n+1)^2 f(n+1) - n^2 f(n) \\ &= \frac{n^2}{(n+1)^2-1} f(n) \\ &= \frac{n}{n+2} f(n) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{f(2)}{f1}\frac{f(3)}{f(2)}\cdots\frac{f(1996)}{f(1995)} &= \frac{1}{3}\frac{2}{4}\cdots\frac{1994}{1996}\frac{1995}{1997} \\ \implies \frac{f(1996)}{f(1)} &= \frac{1}{1996}\frac{2}{1997} \\ \implies f(1996) &= \frac{2}{1997} \end{aligned}$$

**Solution 61**: We have

$$\begin{aligned} a_{n+1} &= a_n \frac{2(n+2)}{n+1} \\ \implies \frac{a_2}{a_1}\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}} = a_n &= 2^{n-2}(n+1) \end{aligned}$$

**Solution 62**: We need to consider the following cases,

$$\begin{cases} x^2-2x-48=0 \text{ if } x\le -16 \text{ has no real roots} \\ x^2-16=0 \text{ if} -16<x<-3 \text{ has one real root } x=-4 \\ x^2-2=0 \text{ if} -3\le x\le 3 \text{ has two real roots} \pm\sqrt{3} \\ x^2-16=0 \text{ if } 3<x<4 \text{ has no real roots} \\ x^2+2x-24=0 \text{ if } x\ge 4 \text{ has one real root } x=4 \end{cases}.$$

From the above, we see that the number of real solutions is 4 and the largest real root is 4 so $l+r$ is **8**.

**Solution 63**: Let the integer removed be $x$. The average of the remaining numbers is

$$\begin{aligned} \frac{n(n+1)}{2(n-1)} - \frac{x}{n-1} &= 40\frac{3}{4} \\ \implies \frac{(n-1)^2 + 3(n-1) + 2}{2(n-1)} - \frac{x}{n-1} &= 40\frac{3}{4} \\ \implies \frac{n-1}{2} + \frac{3}{2} + \frac{1}{n-1} - \frac{x}{n-1} &= 40\frac{3}{4} \\ \implies \frac{n-1}{2} = 40 \wedge \frac{x-1}{n-1} &= \frac{3}{4} \end{aligned}$$

From the above we get $n = 81$ and $x = 61$.

**Solution 64**: Using **Cauchy-Schwarz** inequality, we have

$$\begin{aligned} &(b+c+d+e)^2 \leq \left(b^2+c^2+d^2+e^2\right)\left(1^2+1^2+1^2+1^2\right) \\ &\implies (8-a)^2 \leq 4\left(16-a^2\right) \\ &\implies 0 \leq a \leq \frac{16}{5} \end{aligned}$$

Therefore, maximum value of $a$ is $\frac{16}{5}$ and minimum value is 0.

**Solution 65**: As $a^n - b^n$ is divisible by $a - b$ for all $n$,

$121^n - (-4)^n$ is divisible by $121 - (-4) = 125$.

$1900^n - 25^n$ is divisible by $1900 - 25 = 1875$ which is divisible by 125.

From the above we see that $121^n - 25^n + 1900^n - (-4)^n$ is divisible by 125.

We also have

$$\begin{aligned} 121 \equiv 1 \bmod 8 &\implies 121^n \equiv 1 \bmod 8 \\ 25 \equiv 1 \bmod 8 &\implies 25^n \equiv 1 \bmod 8 \\ 1900 \equiv 4 \bmod 8 &\implies 1900^n \equiv 4^n \bmod 8 \\ -4 \equiv 4 \bmod 8 &\implies (-4)^n \equiv 4^n \bmod 8 \end{aligned}$$

From the above we see that $121^n - 25^n + 1900^n - (-4)^n$ is divisible by 8.

As $121^n - 25^n + 1900^n - (-4)^n$ is divisible by 125 and 8 which are relatively prime to each other, it is divisible by $125 \cdot 8 = 2000$.

**Solution 66**: Using $AM \geq GM$ as all the quantities involved are positive, we have

$$\frac{x^2 + 2xy + 2xy + 4y^2 + z^2 + z^2}{6} \geq \left(x^2 \cdot 2xy \cdot 2xy \cdot 4y^2 \cdot z^2 \cdot z^2\right)^{\frac{1}{6}} = 16$$
$$\implies x^2 + 4xy + 4y^2 + 2z^2 \geq 96$$

Therefore, the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$ is 96.

**Solution 67**: Every square integer is of the form $8k$, $8k + 1$ or $8k + 4$.

It is sufficient to consider the following 6 cases

| Form of $a^2$ | Form of $b^2$ | Form of $a^2 + b^2 - 8c$ |
|---|---|---|
| $8s$ | $8t$ | $8(s + t - c)$ |
| $8s$ | $8t + 1$ | $8(s + t - c) + 1$ |
| $8s + 1$ | $8t + 1$ | $8(s + t - c) + 2$ |
| $8s$ | $8t + 4$ | $8(s + t - c) + 4$ |
| $8s + 1$ | $8t + 4$ | $8(s + t - c) + 5$ |
| $8s + 4$ | $8t + 4$ | $8(s + t - c + 1)$ |

In each of the cases, it is easy to see that $a^2 + b^2 - 8c$ can never be 6.

**Solution 68**: From and we have,

$$(x + y)(ax + by) = 7(x + y)$$
$$\implies ax^2 + by^2 + xy(a + b) = 7(x + y)$$
$$\implies xy(a + b) = 7(x + y) - 49$$

From and we have,

$$(x + y)(ax^2 + by^2) = 49(x + y)$$
$$\implies ax^3 + by^3 + xy(ax + by) + = 49(x + y)$$
$$\implies xy = 7(x + y) - 19$$

From and we have,

$$\begin{aligned}(x+y)(ax^3+by^3) &= 133(x+y)\\ \implies ax^4+by^4+xy(ax^2+by^2) &= 133(x+y)\\ \implies 7xy &= 19(x+y)-58\end{aligned}$$

From , and we have, $xy=\frac{-3}{2}$, $x+y=\frac{5}{2}$ and $a+b=21$.

Therefore the value of $2014(x+y-xy)-100(a+b)=\mathbf{5956}$.

**Solution 69**: Factorizing,we have

$$\begin{aligned}3^{32}-2^{32} &= (3^{16}+2^{16})((3^{16}-2^{16})\\ &= (3^{16}+2^{16})(3^8+2^8)(3^4+2^4)(3^2+2^2)(3^2-2^2)\end{aligned}$$

From **Fermat's Little Theorem**, we have $3^{17-1}-1-(2^{17-1}-1)$ is divisible by 17. Therefore, $(3^{32}-2^{32})$ is divisible by 5, 13 and 97.

**Solution 70**: Completing squares on both sides we get

$$\begin{aligned}(n-3)^2-9 &= \left(m+\frac{1}{2}\right)^2-\frac{1}{4}-10\\ \implies (2m+1)^2-(2n-6)^2 &= 5\\ \Rightarrow (2m+2n-5)(2m-2n+7) &= 1\cdot 5\end{aligned}$$

We get the following sets of simultaneous equations,

$$\begin{aligned}2m-2n=-6 \vee 2m-2n &= -2\\ 2m+2n=10 \vee 2m+2n &= 6\end{aligned}$$

Solving the above, we get $(m,n)=(1,4)$ or $(m,n)=(1,2)$.

**Solution 71**: By $AM \geq GM$, we have

$$\frac{(x+x+y)}{3}\frac{(y+x+y)}{3} \geq (x^2y)^{\frac{1}{3}}(y^2x)^{\frac{1}{3}} = xy$$

Replacing $x+y$ by 1, multiplying both sides by 9 and dividing by $xy$, we get the required inequality.

**Solution 72**: The digits in the hundreds place of the numbers on the tiles have to be $9,8,7$, the digits in the tens place have to be

6, 5, 4 and the digits in the ones place have to be 3, 2, 1. The number of ways in which the tiles can be arranged in $3! \cdot 3! \cdot 3! = \mathbf{216}$ ways.

**Solution 73**: The first time 45 appears in the red list is when the number 99999 appears in the blue list. The numbers 99909, 99918, 99927, 99936, 99945, 99954, 99963, 99972, 99981, 99990 all have the digit sum 36 and appear before 9999 in the blue list so 36 appears atleast 5 times in the red list before 45 appears.

**Solution 74**: Let the sides of rectangle $ABCD$ be $l$ and $b$. The area of the quadrilateral $A'B'C'D'$ is given by $lb + 2 \cdot \frac{1}{2} \cdot k \cdot l \cdot (b + kb) + 2 \cdot \frac{1}{2} \cdot k \cdot b \cdot (l + kl) = lb(1 + 2k + 2k^2)$. Therefore,

$$1 + 2k + 2k^2 = 25$$
$$=> k^2 + k - 12 = 0$$
$$\Rightarrow (k + 4)(k - 3) = 0.$$

As $k > 0$, $k$ has to be $\mathbf{3}$.

**Solution 75**: We need to express 3505 as a sum of powers of 2. Is it natural to look at the binary representation of 3505 which is $110110110001_2$. From the binary representation, it is clear that one of the numbers has to be 1. If the numbers at the vertices are $2^a$, $2^b$ and $2^c$, the numbers on the edges would be $2^a$, $2^b$, $2^c$, $2^{a+b}$, $2^{b+c}$ and $2^{a+c}$ so we would have a one in the $(a+1)^{th}$th place, $(b+1)^{th}$, $(c+1)^{th}$, $(a+b)^{th}$, $(a+c)^{th}$ and $(b+c)^{th}$ place. From the binary representation we see that $a = 3$ and $b = 4$ and $c = 7$ satisy the above condition. So the numbers are $1, 8, 8, 16, 16, 128, 128, 128, 1024, 2048$.

**Solution 76**: Rearranging the terms we get

$$2^x = \frac{2(x+2)}{4-x}.$$

The LHS is always positive, so we have

$$(x+2)(4-x) > 0 \Rightarrow -2 < x < 4.$$

We only have to check whethere they are any solutions to the equation for $x \in \{-1, 0, 1, 2, 3\}$. It is easy to check that $x = 0, x = 1$ and $x = 2$ are the only integer solutions.

**Solution 77**: The sum of the digits is 45 which is a multiple of 9. Each of the three numbers has to be of the form $3k + 1$ or $3k + 2$. To get a number of the form $3k + 1$, we can select one digit of the form $3k$ and two digits of the form $3k + 2$, two digits of the form $3k$ and one of the form $3k + 1$ and two of the form $3k + 1$ and one of the form $3k + 2$. For example the three numbers could be 325, 169 and 478.

**Solution 78**: Let the speeds of Daniel, Marcelo and Gerardo be $d$, $m$ and $g$ m/s respectively. We have

$$\frac{46}{g} = \frac{50}{d} \wedge \frac{200}{g} = \frac{185}{m} \Rightarrow \frac{d}{m} = \frac{1000}{851}.$$

Therefore, Daniel should give Marcelo a headstart of $1000 - 851 = \mathbf{149}$m.

**Solution 79**: We see that number of people in the first generation is $2^1 + 2^0$, in the second generation is $2^2 + 2^1$. The number of people in the $n^{th}$ generation is $2^n + 2^{n-1}$. Therefore the number of people upto the $10^{th}$ generation is

$$\sum_{i=1}^{10} 2^i + 2^{i-1} = 2^{10} - 1 + 2(2^{10} - 1) = 3(2^{10} - 1).$$

**Solution 80**: Let $x = \log_a(b)$. We have $(x - 2)(x - 3) = 0$. We need tuples of the form $(x, x^2)$ and $(y, y^3)$. where $x^2 < 2023$ and $y^3 < 2023$. The total number of tuples is

$$\left\lfloor \sqrt{2023} \right\rfloor + \left\lfloor \sqrt[3]{2023} \right\rfloor = 43 + 11 = \mathbf{54}.$$

**Solution 81**: We have

$$16\beta < 37\alpha \wedge 16\alpha < 7\beta \Rightarrow 7\beta < 23\alpha.$$

Therefore, the smallest value of $\beta$ is **23**.

**Solution 82**: If $n = 2k$ is the number of sides of the polygon, we need $\frac{(2k-2)180}{2k}$ to be an integer for $n$ to belong to $X$. As $(k - 1)$

and $k$ are relatively prime to each other, $k$ divides 180. The factors of 180 greater than 1 are 2,3,4,5,6,9,10,12,15,18,20,30,36,45,60 and 90. Therefore the number of elements in $X$ is $\mathbf{16}$. Another way to calculate the number of prime factors is to use the prime factorization of $180 = 2^2 \cdot 3^2 \cdot 5$. Therefore, the number of factors of 180 are $(2+1)(2+1)(1+1) = 18$. These include 1 and 180, excluding them we get 16 factors, so the number of elements in $X$ is $\mathbf{16}$.

**Solution 83**: Let $T(n)$ be the number of ways of tiling a $2 \times n$ rectangle with a $2 \times 1$ domino. We have $T(1) = 1$ and $T(2) = 2$. When we start tiling a $2 \times n$ rectangle with a vertical $2 \times 1$ domino, the remaining $2 \times (n-1)$ rectangle can be tiled $T(n-1)$ ways. When we start tiling a $2 \times n$ rectangle with a horizontal $2 \times 1$ domino, the rest of the rectangle can be tiled in $T(n-2)$ ways. Therefore, we have $T(n) = T(n-1) + T(n-2)$ ways of tiling a $2 \times n$ rectangle. When we use a $2 \times 2$ tile, we can tile $2 \times 7$ rectangle by putting the $2 \times 2$ tile in one of the 6 positions. In the first position, we can tile the rectangle in $T(5)$ ways, in the second position the rectangle can be tiled in $T(1) \cdot T(4)$ ways and in the third position we have $T(2) \cdot T(3)$ ways. Using symmetry, the total number of ways of tiling a $2 \times 7$ rectangle with one $2 \times 2$ tile is $2(T(5) + T(1) \cdot T(4) + T(2) \cdot T(3))$. In the case of $2 \times 7$ rectangle, the total number of ways of tiling is $2(8 + 1 \cdot 5 + 2 \cdot 3) + T(7) = 2 \cdot 19 + 21 = \mathbf{59}$.

**Solution 84**: We have $p_1 + p_2 + p_3 = 0 \vee p_1 = p_2 = p_3$. $p_1 + p_2 + p_3 = 0 \Rightarrow 36 + 14a + 6b + 3c = 0$ which is not possible as $c$ is odd, therefore $p_1 = p_2 = p_3$. We now have the following equations,

$$1 + a + b + c = 8 + 4a + 2b + c \Rightarrow 3a + b = -7$$
$$1 + a + b + c = 27 + 9a + 3b + c \Rightarrow 4a + b = -13.$$

$$p_2 + 2p_1 - 3p_0 = 10 + 6a + 4b = 10 + 6(-6) + 4 \cdot 11 = \mathbf{18}.$$

**Solution 85**: We have,

$$x + y + z = 2$$
$$xy + yz + zx = \frac{(x+y+z)^2 - (x^2+y^2+z^2)}{2} = \frac{2^2 - 3}{2} = \frac{1}{2}$$
$$xyz = 4.$$

Using Vieta's formulas, we see that $x, y$ and $z$ can be considered to be roots of the equation

$$u^3 - 2u^2 + \frac{u}{2} - 4 = 0 \Rightarrow 2u^3 - 4u^2 + u - 8 = 0.$$

The expression,

$$S = \frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}$$
$$= \frac{1}{(1-x)(1-y)} + \frac{1}{(1-y)(1-z)} + \frac{1}{(1-z)(1-z)}.$$

When $z = \frac{1}{1-u}$, the expression $S$ can be considered the product of roots of the equation

$$2\left(1 - \frac{1}{z}\right)^3 - 4\left(1 - \frac{1}{z}\right)^2 + \left(1 - \frac{1}{z}\right) - 8 = 0$$
$$\Rightarrow 9z^3 - z^2 - 2z + 2 = 0$$

taken two at a time. Therefore $S = -\mathbf{\frac{2}{9}}$ using Vieta's formula.

**Solution 86**: The number of trailing zeros in a factorial $n!$ is determined by the number of pairs of factors 2 and 5 in the numbers from 1 to $n$. Because factors of 2 are much more common than factors of 5, the number of trailing zeros is really determined by the number of factors of 5. For any given number $n$, the number of factors of 5 in $n!$ can be calculated by repeatedly dividing $n$ by 5 until the result is less than 5, and adding up the quotients. This is because when you divide $n$ by 5, you count all the numbers that are divisible by 5. But some numbers (like 25, 50, etc.) have more than one factor of 5. You count these extra factors when you divide by $5^2 = 25$, and so on. The highest power of 5 in 365! is

$$\left\lfloor \frac{365}{5} \right\rfloor + \left\lfloor \frac{365}{5^2} \right\rfloor + \left\lfloor \frac{365}{5^3} \right\rfloor = 73 + 14 + 2 = 89.$$

**Solution 87**: Let $R(n)$ be the number of rectangles in a staircase of height $n$. When a new row is added to the staircase of height $n-1$, the number of horizontal rectangles increases by $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ and the number of vertical rectangles increases by $1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2}$. Therefore we have the recurrence $R(n) = R(n-1) + n^2$. We need $R(11) = 1 + 2^2 + \ldots + 11^2 = \mathbf{506}$.

**Solution 88**: We have $t_n = (n + 1 - 1)n! = (n + 1)! - n!$, therefore the required sum is $\sum_{i=1}^{n} t_n = (n + 1)! - 1$.

**Solution 89**: We have

$$t_n = \frac{n + 1 - 1}{(n + 1)!} = \frac{1}{n!} - \frac{1}{(n + 1)!}.$$

Therefore, the required sum is $\sum_{i=1}^{n} t_n = 1 - \frac{1}{(n+1)!}$ .

**Solution 90**: The above product contains $2n$ terms. When $n$ is odd, there are $n$ terms that are **even** starting with $n + 1$ and ending with $2n$. When $n$ is even, there are $n - 1$ terms that are **even** starting with $n + 2$ and ending with $2n$. There is an extra factor of 2 in $2n$ when $n$ is even. Therefore the product under consideration is divisible by $2^n$.

**Solution 91**: Every divisor of the HCF of the two number is a common factor. The HCF of $10^{40} = 2^{40} \cdot 5^{40}$ and $20^{30} = 2^{60} \cdot 5^{30}$ is $2^{40} \cdot 5^{30}$. The number of divisors of the HCF is $(40 + 1) \cdot (30 + 1) = 1271$.

**Solution 92**: If a natural number $n$ has as its prime factorization,

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

where the $p_i$'s are distinct primes and the $k_i$'s are positive integers, then the number of positive divisors of $n$ is given by $\prod_{i=1}^{r} (k_{i+1})$. When $n$ is a square, each $k_i$ is even which implies $k_i + 1$ is odd for $i \in \{1, 2, \ldots, r\}$. Therefore the number of positive divisors of $n$ is odd.

**SOLUTION 93**: A parallelogram can be formed by choosing any two lines out of $p$ lines in the horizontal direction and any two lines out of $q$ lines in the vertical direction. Therefore the number of parallelograms is $\binom{p}{2} \cdot \binom{q}{2}$.

**SOLUTION 94**: Divide the equilateral triangle into 4 smaller equilateral triangles, each of side length $\frac{1}{2}$. There is at least one triangle that has 2 points and the distance between them has to be less than $\frac{1}{2}$.

**SOLUTION 95**: Divide the circle into $n$ equal sectors. There is one sector that has at least 2 points. The angle of each sector is $\frac{2\pi}{n}$. The distance between any two points on the part of the circumference enclosed by a sector is less than the max chord length in the sector which is $2\sin\frac{\pi}{n}$.

**SOLUTION 96**: Divide the square into 4 equal squares of side length $\frac{1}{2}$ , there is one square with at least 3 points. The maximum area of the triangle formed by three points within the small square is half the area of the square which is $\frac{1}{8}$.

**SOLUTION 97**: All natural numbers can be partitioned into three sets, the first set contains all numbers of the form $3k$, the second contains all numbers of the form $3k + 1$ and the third numbers of the form $3k + 2$. If at least three numbers are chosen from the same set, then the sum of any three of those numbers is divisible by 3. The table below lists all cases where less than 3 numbers are chosen from each set:

| 3k | 3k+1 | 3k+2 |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 2 | 1 |
| 2 | 1 | 2 |

The table indicates the number of elements chosen from each set

It is easy to see that in each of the cases, there are three numbers whose sum is divisible by 3.

**Solution 98**: A number when divided by $n$ leaves a remainder $r \in \{0, 1, ..., n-1\}$. When you have $n+1$ numbers, there are at least two numbers, that leave the same remainder when divided by $n$. The difference of those two number is also divisible by $n$.

**Solution 99**: Divide the set $A$ into two sets, one containing all even numbers and one containing all odd numbers. WLOG we can assume that there are $k$ even integers and $k+1$ odd integers. For the product to be odd, the $k+1$ odd integers must be mapped to $k+1$ even integers which is impossible, therefore the product is even.

**Solution 100**: Every integer $n \geq 1$ can be uniquely written in the form $n = 2^a b$ where $b$ is its greatest odd factor. Using this idea, we take the set $\{1, 2, 3, ..., 2n\}$ and replace each element with its greatest odd factor. There are only $n$ odd numbers less than $2n$ so there are only $n$ choices for this factor. Thus, by the pigeonhole principle, when we pick $n+1$ numbers from the set, some two of them must have the same greatest odd factor. These two numbers can be written as $2^a b$ and $2^c b$, so clearly the smaller divides the larger.